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Converse Lagrange Theorem Orders and Supersolvable Orders

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Abstract

For finite groups, we investigate both converse Lagrange theorem (CLT) orders and supersolvable (SS) orders, and see that the latter form a proper subset of the former. We focus on the difference between these two sets of orders, reformulate the work of earlier authors algorithmically, and construct a computer program to enumerate such NSS-CLT orders. We establish several results relating to NSS and CLT orders and, working from our computer-generated data, propose a pair of conjectures and obtain a complete characterization of the most common form of NSS-CLT order.

1 Introduction

Throughout this paper, we consider only *finite* groups, and we begin by recalling one of the most fundamental results in the area, the famous theorem of Lagrange:

Theorem 1. *If G is a group and H is a subgroup of G, then $|H|$ is a divisor of $|G|$.*

However, the converse of this result is false; for example, A_4, the alternating group on four symbols, which has order 12, has no subgroup of order 6 [2].

If G is a group which has a subgroup of order d, for *every* divisor d of $|G|$, then G is a *converse Lagrange theorem* (CLT) group; otherwise, G is a *non-converse Lagrange theorem* (NCLT) group. For example, S_4, the symmetric group on four symbols, is CLT, since it has order 24 and has subgroups of order 1, 2, 3, 4, 6, 8, 12, and 24, all the divisors of 24. As mentioned above, A_4 is an NCLT group.

If *every* group of order n is CLT, then n is a *CLT order*; otherwise, n is an *NCLT order*. For example, 16 is a CLT order, since every group of order 16 has subgroups of order 1, 2, 4, 8, and 16. On the other hand, 24 is an NCLT order, because, although S_4 is a CLT group, another group $SL(2,3)$ of order 24 has no subgroup of order 12 [2].

We also consider the following concept. A group G is *supersolvable*, or *supersoluble*, (SS) if it has a series of subgroups, each normal in the next:

$$\{1\} = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_{k-1} \triangleleft H_k = G$$

satisfying both of the following conditions:

- $H_i \triangleleft G$, for each $0 \leq i \leq k$
- H_i/H_{i-1} is cyclic, for each $1 \leq i \leq k$.

For example, D_4, the group of rigid motions of a square, is SS, whereas the group S_4 is NSS (non-supersolvable).

If *every* group of order n is SS, then n is an *SS order*; otherwise, n is an *NSS order*. For example, 8 is an SS order, whereas 24 is an NSS order since S_4 is an NSS group.

The concepts of CLT groups and SS groups are linked by the following result, which has been proved by several authors (see Bray [3], for example):

Theorem 2. *Every SS group is CLT.*

It follows immediately that *every SS order is a CLT order*; restating this in an equivalent but more useful form for what follows, we have

Theorem 3. *Every NCLT order is an NSS order.*

The converse of Theorem 2 is false. For example, S_4 is CLT yet also NSS. Nevertheless, the possibility remains that the NCLT *orders* and the NSS *orders* coincide. As just seen, S_4, of order 24, is CLT but NSS; however, there exists another group of order 24, $SL(2,3)$, which is NCLT and NSS, making the integer 24 both an NSS order and an NCLT order. However, we shall shortly see that the NCLT orders and the NSS orders do *not* coincide, and so the converse of Theorem 3 is also false.

Curran [5] has found all the NCLT *groups* with order less than 100, but a full structure theorem for the rather complicated class of all NCLT groups currently seems out of reach. However, the NCLT *orders* have been completely determined by Berger [1], although this classification is quite involved, consisting of five different sets of numerical conditions relating to the divisors of the group order and the congruences which they must satisfy. Likewise, the NSS *orders* have been determined by Pazderski [10], and while their classification is somewhat less complicated that that of the NCLT orders, it is still quite intricate (Hughes [8] presents a simpler formulation, and in English rather than in the German of Pazderski's paper).

2 NSS-CLT orders

We have reformulated these classifications of Berger and Pazderski / Hughes algorithmically and implemented them on a computer, so as to view and compare NSS[1] and NCLT orders.

Inspection of Table 1 gives rise to several results and conjectures, as presented below.

Theorem 4. *The NCLT orders form a proper subset of the NSS orders.*

Proof. Theorem 3 establishes inclusion, and the number 224 from Table 1 below, being an NSS order but not an NCLT order, shows that this inclusion is proper. □

The NSS-CLT orders (numbers which are NSS orders but not NCLT orders) form the main focus of this paper. The number $224 = 2^5 \cdot 7$ is in fact the smallest NSS-CLT order; the next such order is $2464 = 2^5 \cdot 7 \cdot 11$, while the first such odd order is $3159 = 3^5 \cdot 13$.

Theorem 5. *Every multiple of an NSS order is itself an NSS order.*

Proof. Let k be a positive integer and n an NSS order. Choose any NSS group of order n, and consider its direct product with the cyclic group C_k. The resulting group has order kn, and is clearly NSS, as it has an NSS subgroup; this follows from the fact that every subgroup of an SS group is SS. Thus kn is an NSS order, as claimed. □

However, the situation is different in the case of NCLT orders:

Theorem 6. *A multiple of an NCLT order need not be an NCLT order.*

Proof. From Table 1 below, a minimal counterexample is given by the CLT order $224 = 2^5 \cdot 7$, which is a multiple of the NCLT order $56 = 2^3 \cdot 7$. □

The minimal such odd order is the CLT order $3159 = 3^5 \cdot 13$, which is a multiple of the NCLT order $351 = 3^3 \cdot 13$. For completeness, we also note

Theorem 7. *A multiple of an NSS-CLT order need not be an NSS-CLT order.*

Proof. We have $2 \cdot 224 = 448$; but 224 is an NSS-CLT order, while 448 is not. □

[1] NSS orders also appear as sequence A066085 in the *On-Line Encyclopedia of Integer Sequences* [11].

NSS Order	NCLT?	NSS Order	NCLT?	NSS Order	NCLT?
$12 = 2^2 \cdot 3$	√	$363 = 3 \cdot 11^2$	√	$5376 = 2^8 \cdot 3 \cdot 7$	√
$24 = 2^3 \cdot 3$	√	⋯		⋯	
$36 = 2^2 \cdot 3^2$	√	$448 = 2^6 \cdot 7$	√	$5600 = 2^5 \cdot 5^2 \cdot 7$	√
$48 = 2^4 \cdot 3$	√	⋯		⋯	
$56 = 2^3 \cdot 7$	√	$672 = 2^5 \cdot 3 \cdot 7$	√	$5824 = 2^6 \cdot 7 \cdot 13$	√
$60 = 2^2 \cdot 3 \cdot 5$	√	⋯		⋯	
$72 = 2^3 \cdot 3^2$	√	$896 = 2^7 \cdot 7$	√	$6048 = 2^5 \cdot 3^3 \cdot 7$	√
$75 = 3 \cdot 5^2$	√	⋯		⋯	
$80 = 2^4 \cdot 5$	√	$1120 = 2^5 \cdot 5 \cdot 7$	√	$6272 = 2^7 \cdot 7^2$	√
$84 = 2^2 \cdot 3 \cdot 7$	√	⋯		⋯	
$96 = 2^5 \cdot 3$	√	$1344 = 2^6 \cdot 3 \cdot 7$	√	$6318 = 2 \cdot 3^5 \cdot 13$	×
$108 = 2^2 \cdot 3^3$	√	⋯		⋯	
$112 = 2^4 \cdot 7$	√	$1568 = 2^5 \cdot 7^2$	√	$6496 = 2^5 \cdot 7 \cdot 29$	×
$120 = 2^3 \cdot 3 \cdot 5$	√	⋯		⋯	
$132 = 2^2 \cdot 3 \cdot 11$	√	$1792 = 2^8 \cdot 7$	√	$6720 = 2^6 \cdot 3 \cdot 5 \cdot 7$	√
$144 = 2^4 \cdot 3^2$	√	⋯		⋯	
$150 = 2 \cdot 3 \cdot 5^2$	√	$2016 = 2^5 \cdot 3^2 \cdot 7$	√	$6944 = 2^5 \cdot 7 \cdot 31$	√
$156 = 2^2 \cdot 3 \cdot 13$	√	⋯		⋯	
$160 = 2^5 \cdot 5$	√	$2240 = 2^6 \cdot 5 \cdot 7$	√	$7168 = 2^{10} \cdot 7$	√
$168 = 2^3 \cdot 3 \cdot 7$	√	⋯		⋯	
$180 = 2^2 \cdot 3^2 \cdot 5$	√	$2464 = 2^5 \cdot 7 \cdot 11$	×	$7392 = 2^5 \cdot 3 \cdot 7 \cdot 11$	√
$192 = 2^6 \cdot 3$	√	⋯		⋯	
$196 = 2^2 \cdot 7^2$	√	$2688 = 2^7 \cdot 3 \cdot 7$	√	$7616 = 2^6 \cdot 7 \cdot 17$	√
$200 = 2^3 \cdot 5^2$	√	⋯		⋯	
$204 = 2^2 \cdot 3 \cdot 17$	√	$2912 = 2^5 \cdot 7 \cdot 13$	×	$7840 = 2^5 \cdot 5 \cdot 7^2$	√
$216 = 2^3 \cdot 3^3$	√	⋯		⋯	
$224 = 2^5 \cdot 7$	×	$3136 = 2^6 \cdot 7^2$	√	$8064 = 2^7 \cdot 3^2 \cdot 7$	√
$225 = 3^2 \cdot 5^2$	√	⋯		⋯	
$228 = 2^2 \cdot 3 \cdot 19$	√	$3159 = 3^5 \cdot 13$	×	$8288 = 2^5 \cdot 7 \cdot 37$	×
$240 = 2^4 \cdot 3 \cdot 5$	√	⋯		⋯	
$252 = 2^2 \cdot 3^2 \cdot 7$	√	$3808 = 2^5 \cdot 7 \cdot 17$	×	$8512 = 2^6 \cdot 7 \cdot 19$	√
$264 = 2^3 \cdot 3 \cdot 11$	√	⋯		⋯	
$276 = 2^2 \cdot 3 \cdot 23$	√	$4032 = 2^6 \cdot 3^2 \cdot 7$	√	$8736 = 2^5 \cdot 3 \cdot 7 \cdot 13$	√
$280 = 2^3 \cdot 5 \cdot 7$	√	⋯		⋯	
$288 = 2^5 \cdot 3^2$	√	$4256 = 2^5 \cdot 7 \cdot 19$	×	$8960 = 2^8 \cdot 5 \cdot 7$	√
$294 = 2 \cdot 3 \cdot 7^2$	√	⋯		⋯	
$300 = 2^2 \cdot 3 \cdot 5^2$	√	$4480 = 2^7 \cdot 5 \cdot 7$	√	$9184 = 2^5 \cdot 7 \cdot 41$	×
$312 = 2^3 \cdot 3 \cdot 13$	√	⋯		⋯	
⋯		$4704 = 2^5 \cdot 3 \cdot 7^2$	√	$9408 = 2^6 \cdot 3 \cdot 7^2$	√
$351 = 3^3 \cdot 13$	√	⋯		⋯	
$360 = 2^3 \cdot 3^2 \cdot 5$	√	$4928 = 2^6 \cdot 7 \cdot 11$	√	$9632 = 2^5 \cdot 7 \cdot 43$	×
		⋯		⋯	
		$5152 = 2^5 \cdot 7 \cdot 23$	×	$9856 = 2^7 \cdot 7 \cdot 11$	√
		⋯		⋯	

Table 1: Some NSS orders and their NCLT status

We note that the number 224 occurring in Theorems 4 and 6 can be derived from another source — the following theorem of Struik [12] — which, however, makes no mention of minimality:

Theorem 8. *Let p and q be primes, such that $q|(p-1)$ and such that f, the exponent of $q \pmod p$, is odd. Then for each $1 \leq m < q$, the number $q^{2f-1}\,p^m$ is an NSS-CLT order.*

Choosing $p = 7$ and $q = 2$ gives $q|(p-1)$ and $f = 3$, which is odd. The only valid choice for m is 1. Thus $q^{2f-1}\,p^m = 2^5 \cdot 7^1 = 224$ is an NSS-CLT order, as already seen.

Note that Theorem 8 does not generate *all* NSS-CLT orders; for example, the second smallest NSS-CLT order is $2464 = 2^5 \cdot 7 \cdot 11$, but this has three distinct prime factors.

We also recall the following result of Humphreys and Johnson [9]:

Theorem 9. *Every CLT group of cubefree order is an SS group.*

The cubefree condition is necessary, as the NSS-CLT group S_4, of order $24 = 2^3 \cdot 3$, shows. Moving now from groups to orders in Theorem 9, and presenting the contrapositive to better align with our general exposition, yields

Theorem 10. *Every cubefree NSS order is an NCLT order.*

Our computer-generated list of NSS-CLT orders, which extends Table 1 above, contains the number $453789 = 3^3 \cdot 7^5$, giving

Theorem 11. *The cubefree condition in Theorem 10 is necessary.*

The following well-known result of Deskins [6] will be useful in the sequel:

Theorem 12. *If G is CLT and every subgroup of G is CLT, then G is SS.*

Corollary 13. *If n is a CLT order and every proper divisor of n is a CLT order, then n is an SS order.*

We now introduce the concepts of NCLT orders and NSS orders being *primitive*.

Definition 14. A number is a *primitive NCLT order* if it is an NCLT order but none of its proper divisors is an NCLT order.

For example, Table 1 above shows that the NCLT orders 12, 56, and 75 are primitive, whereas $24 = 2 \cdot 12$ is not.

Definition 15. A number is a *primitive NSS order* if it is an NSS order but none of its proper divisors is an NSS order.

For example, Table 1 above shows that the NSS orders 12, 56, and 75 are primitive, whereas $224 = 4 \cdot 56$ is not.

Theorem 16. *The primitive NCLT orders and the primitive NSS orders coincide.*

Proof. Let n be a primitive NCLT order. From Theorem 3 we know that n is an NSS order, so we need to show that if d is a proper divisor of n, then every group G of order d is SS. By hypothesis, G is CLT; moreover, every subgroup of G, whose order divides d and thus n, is also CLT. By Theorem 12, G is SS, and so n is a primitive NSS order.

Conversely, let n now be a primitive NSS order. Every proper divisor of n is an SS order, and thus, from Theorem 2, a CLT order. If n itself were a CLT order, then by Corollary 13, n would be an SS order, a contradiction. So n must be an NCLT order. In fact, n must be a *primitive* NCLT order, since otherwise there would exist an NCLT group H whose order is a proper divisor of n; but from Theorem 2, H is an NSS group, contradicting the assumption that n is a primitive NSS order. □

Lemma 17. *For each prime $p > 224$, the number $224p$ is an NSS-CLT order.*

Proof. From Table 1 we observe that 224 is an NSS order, so it then follows from Theorem 5 that $224p$ is an NSS order.

To show that $224p$ is a CLT order for each prime $p > 224$, let G be an arbitrary group of order $n = 224p$ and let n_p be the number of Sylow p-subgroups of G. By Sylow's theorems, n_p must be a divisor of n, and $n_p \equiv 1 \pmod{p}$. Now the divisors of n are

$$\begin{array}{cccccccccccc} 1, & 2, & 4, & 8, & 16, & 32, & 7, & 14, & 28, & 56, & 112, & 224, \\ p, & 2p, & 4p, & 8p, & 16p, & 32p, & 7p, & 14p, & 28p, & 56p, & 112p, & 224p. \end{array}$$

The condition $n_p \equiv 1 \pmod{p}$ eliminates the second line, and the fact that $p > 224$ leaves $n_p = 1$ as the only possibility. Again, from Sylow's theorems it follows that G has a unique Sylow p-subgroup G_p, and that $G_p \triangleleft G$. Clearly, $|G_p| = p$.

Now $|G/G_p| = 2^5 \cdot 7$. By Burnside's famous "pq theorem" [4, ch. XV], G/G_p is solvable. Since G_p, being a p-group, is also solvable, it then follows that G itself is solvable.

By Hall's theorem [7], G has a (Hall) subgroup H of order $2^5 \cdot 7$. As $2^5 \cdot 7$ is a CLT order (see Table 1), H has subgroups of every order dividing $2^5 \cdot 7$; thus also:

$$G \text{ has subgroups of orders } 1, 2, 4, 8, 16, 32, 7, 14, 28, 56, 112, 224. \tag{1}$$

As noted above, $|G/G_p| = 2^5 \cdot 7$, which is a CLT order, so G/G_p has subgroups of orders 1, 2, 4, 8, 16, 32, 7, 14, 28, 56, 112, 224. But each subgroup of G/G_p has the form K/G_p, for K a subgroup of G, and since $|K/G_p| = |K|/p$, it follows that:

$$G \text{ has subgroups of orders } p, 2p, 4p, 8p, 16p, 32p, 7p, 14p, 28p, 56p, 112p, 224p. \tag{2}$$

Combining (1) and (2) shows that G is a CLT group, and so $224p$ is a CLT order. □

Note from Table 1 that this result does *not* hold for any of the primes $p = 2, 3, 5, 7, 31$. But inspecting an extended version of this table, as generated by our computer program, reveals that these are the *only* exceptions in the range $p \leq 224$, giving

Theorem 18. *The number $224p$ is an NSS-CLT order, for all primes $p \neq 2, 3, 5, 7, 31$.*

Corollary 19. *There are infinitely many NSS-CLT orders.*

3 Two conjectures and a characterization theorem

Table 1, generated by our computer program, includes the 12 NSS-CLT orders up to 10,000. We performed an extended run of this program to produce a list of all NSS-CLT orders up to 1,000,000,000. Inspecting the resulting list gave rise to a pair of conjectures, described below, along with a remarkable characterization of the most common form of NSS-CLT order.

We first found the percentage of integers, less than or equal to a given limit, which are NSS-CLT orders. Letting

$$f_1(n) \;=\; \text{the number of NSS-CLT orders up to } n$$

we obtain the following:

n	$f_1(n)$	$f_1(n)/n$
10,000	12	0.120000%
100,000	107	0.107000%
1,000,000	1,094	0.109400%
10,000,000	10,889	0.108890%
100,000,000	108,906	0.108906%
1,000,000,000	1,089,086	0.108909%

Table 2: Number of NSS-CLT Orders

This leads us to propose

Conjecture 20. The proportion of positive integers which are NSS-CLT orders converges to a non-zero constant, whose value is approximately 0.1089%.

It is a little surprising to us that a *fixed* proportion of orders appear to be NSS-CLT.

Looking back at Table 1, observe that no fewer than 10 of its 12 NSS-CLT orders are multiples[2] of 224, the smallest such order. We investigated this further. Letting

$$f_2(n) \;=\; \text{the number of NSS-CLT orders up to } n \text{ which are multiples of } 224$$

we extend Table 2 to obtain the following:

n	$f_2(n)$	$f_1(n)$	$f_2(n)/f_1(n)$
10,000	10	12	83.333%
100,000	95	107	88.785%
1,000,000	961	1,094	87.843%
10,000,000	9,584	10,889	88.015%
100,000,000	95,846	108,906	88.008%
1,000,000,000	958,550	1,089,086	88.014%

Table 3: Number of NSS-CLT Orders which are multiples of 224

[2]We consider 224 itself to be a *multiple* of 224.

Conjecture 21. The proportion of NSS-CLT orders which are multiples of 224 converges to a non-zero constant, whose value is approximately 88%.

Again, it is a little surprising to us that a *fixed* proportion of NSS-CLT orders — and a large one at that — appear to be multiples of 224. Of course, from Theorem 18 we already know that $224p$ is an NSS-CLT order, for all primes $p \neq 2, 3, 5, 7, 31$.

Although the values of n chosen for Table 2 and Table 3 are all consecutive powers of 10, we also generated data for several other random intermediate values of n, and the results are entirely consistent with the above tables.

Due to their relative abundance, we then looked more closely at those NSS-CLT orders which are proper multiples of 224. Besides those of the form $224p$ (see Theorem 18), we have

$32032 = 224 \cdot 11 \cdot 13$	$91168 = 224 \cdot 11 \cdot 37$	$1076768 = 224 \cdot 11 \cdot 19 \cdot 23$
$41888 = 224 \cdot 11 \cdot 17$	$97888 = 224 \cdot 19 \cdot 23$	$1138592 = 224 \cdot 13 \cdot 17 \cdot 23$
$46816 = 224 \cdot 11 \cdot 19$	$101024 = 224 \cdot 11 \cdot 41$	$1185184 = 224 \cdot 11 \cdot 13 \cdot 37$
$49504 = 224 \cdot 13 \cdot 17$	$\cdots$	$1214752 = 224 \cdot 11 \cdot 17 \cdot 29$
$55328 = 224 \cdot 13 \cdot 19$	$544544 = 224 \cdot 11 \cdot 13 \cdot 17$	$1272544 = 224 \cdot 13 \cdot 19 \cdot 23$
$56672 = 224 \cdot 11 \cdot 23$	$608608 = 224 \cdot 11 \cdot 13 \cdot 19$	$1313312 = 224 \cdot 11 \cdot 13 \cdot 41$
$66976 = 224 \cdot 13 \cdot 23$	$736736 = 224 \cdot 11 \cdot 13 \cdot 23$	$\cdots$
$71456 = 224 \cdot 11 \cdot 29$	$795872 = 224 \cdot 11 \cdot 17 \cdot 19$	$10346336 = 224 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
$72352 = 224 \cdot 17 \cdot 19$	$928928 = 224 \cdot 11 \cdot 13 \cdot 29$	$12524512 = 224 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$84448 = 224 \cdot 13 \cdot 29$	$940576 = 224 \cdot 13 \cdot 17 \cdot 19$	$13997984 = 224 \cdot 11 \cdot 13 \cdot 19 \cdot 23$
$87584 = 224 \cdot 17 \cdot 23$	$963424 = 224 \cdot 11 \cdot 17 \cdot 23$	$\cdots$

Table 4: NSS-CLT non-prime multiples of 224

The pattern emerging here is surprising and really quite remarkable. For ease of reference, we temporarily introduce the following phrase:

Definition 22. A *special number* is any integer of the form

$$224\, p_1\, p_2 \cdots p_k$$

where $k \geq 0$, each p_i is a distinct prime, and no p_i equals 2, 3, 5, 7, or 31.

Theorem 23. *The NSS-CLT multiples of 224 are precisely the special numbers.*

Proof. Note at the outset that, by Theorem 5, every multiple of the NSS order 224 is itself an NSS order. So we need only show that the CLT multiples of 224 are the special numbers.

Our proof uses both the characterization of CLT orders given in Berger [1], and the notation of that paper.

We first show that every CLT multiple of 224 is a special number. Accordingly, let n be any CLT multiple of 224 ($= 2^5\, 7$) and let

$$2^5\, 7\, p_1\, p_2 \cdots p_k$$

be its prime factorization, for some $k \geq 0$. We show that the primes p_i are pairwise distinct, and that no p_i equals 2, 3, 5, 7, or 31.

Let q be any odd prime, and express n as $n = \ell\, 2^a q^b$, where $2 \nmid \ell$ and $q \nmid \ell$ (so a and b are the highest powers of 2 and q, respectively, in n). Since $a \geq 5$, Proposition 3.6 [1] gives $b = 0$ or $b = 1$. Thus the primes p_i are pairwise distinct. Moreover, no p_i equals 7.

Now express n as $n = \ell\, 7^1 2^b$, where $7 \nmid \ell$ and $2 \nmid \ell$. Since the exponent of 2 (mod 7) is 3, Proposition 3.5 [1] gives $b \in \{0, 1, 2, 5\}$. In particular, $b \leq 5$, so no p_i can equal 2.

Finally, express n as $n = \ell\, p\, 2^b$, where $b = 5$, p is an odd prime, $p \nmid \ell$, and $2 \nmid \ell$. Letting d denote the exponent of 2 (mod p), consider the possibilities:

$$\begin{aligned} p = \ \ 3 &: \ d = 2, \text{ but } b = 5 \notin \{0, 1\} \\ p = \ \ 5 &: \ d = 4, \text{ but } b = 5 \notin \{0, 1, 2, 3\} \\ p = 31 &: \ d = 5, \text{ but } b = 5 \notin \{0, 1, 2, 3, 4, 9\} \end{aligned}$$

and by Proposition 3.5 [1], n cannot be a CLT order in any of these three cases. Thus, no p_i equals 3, 5, or 31, completing the proof that every CLT multiple of 224 is a special number.

We now show that every special number, clearly being a multiple of 224, is a CLT order, by showing that it is "good" [1]. Accordingly, consider any special number

$$n \ = \ 2^5\, 7\, p_1\, p_2 \cdots p_k \ = \ \textstyle\prod_{q \in Q} q^{e(q)}$$

where $Q = \{2, 7, p_1, \cdots, p_k\}$, $e(2) = 5$, $e(7) = 1$, and $e(p_i) = 1$ for $1 \leq i \leq k$. We first show that $e(q) \in \mathscr{S}(m, q)$ for each divisor m of n and each $q \in Q$ (Berger [1] defines the set $\mathscr{S}$).

If m is composite, then $\mathscr{S}(m, q) = \mathbb{N} \setminus \{0\}$ for each q, so $e(q) \in \mathscr{S}(m, q)$. Otherwise, m is a prime divisor of n, so in fact $m \in Q$. For $q = 2$, we have the following cases; again, Berger [1] defines the sets $\mathscr{I}$ and $\mathscr{I}'$:

$$\begin{aligned} \mathscr{S}(\ 2, 2\ ) \ &= \ \mathbb{N} \setminus \{0\} \\ \mathscr{S}(\ 7, 2\ ) \ &= \ \mathscr{I}(3) \ = \ \{1, 2, 5\} \\ \mathscr{S}(\ p_i, 2\ ) \ &= \ \mathscr{I}(d) \text{ or } \mathscr{I}'(d),\ d \geq 6 \ \text{ (check } p_i = 11, 13, 17, 19, 23, 29, \text{ and } p_i \geq 37) \end{aligned}$$

and in each case, $e(2) = 5 \in \mathscr{S}(m, 2)$. For all other cases ($q \in Q, q \neq 2$), we have $e(q) = 1$, and since 1 always belongs to the set $\mathscr{S}(m, q)$, this gives $e(q) \in \mathscr{S}(m, q)$.

We next show that $e(q) \in \mathscr{S}(r, p^u, q)$ for each prime r and p and positive integer u such that rp^u divides n, and each $q \in Q$.

For $q = 2$, there are no primes r and p for which $rp \mid q - 1$, giving $\mathscr{S}(r, p^u, q) = \mathbb{N} \setminus \{0\}$ and thus $e(q) = 5 \in \mathscr{S}(r, p^u, q)$. For all other cases ($q \in Q, q \neq 2$), we again have $e(q) = 1$, and since 1 always belongs to the set $\mathscr{S}(r, p^u, q)$, this gives $e(q) \in \mathscr{S}(r, p^u, q)$.

So the special number n is "good", and by Theorem 1.1 [1], is a CLT order. □

Theorem 23 greatly extends Theorem 18; nonetheless, we opted to retain the earlier result in this paper due to its significantly different and simpler proof.

We conclude by noting, with appreciation, the immense contribution of the computer to the above work. To *manually* determine the NSS-CLT status of even a single integer, using the results of Berger and Pazderski / Hughes, would be both tedious and error-prone. Yet in under 16 hours on a modest desktop, our program had done so for the first billion positive integers. The resulting list of NSS-CLT orders was useful for several of our theorems, and essential for both of our conjectures. In particular, the surprising pattern of Theorem 23 would not have been apparent without such a very extensive list of NSS-CLT orders. Thus,

in these explorations, as in several other areas of discrete mathematics, the computer proves to be a valuable tool and a most helpful assistant.

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